ORCS Summary Paul Disbeschl

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June 2019, version 0.37

Introduction

This document was written to help people understand ORCS. **In its current form, it assumes the reader has full access to the slides.** The goal of this document is to explain parts of the course clearly and concisely with examples. Important parts are explained in detail, others are summarized. It also provides quite a bit of information on Frank's part. I've included examples from the lectures, from the book and elsewhere. There are also YouTube videos explaining things step by step. Enjoy! - Paul Disbeschl, June 2019.
If you're frantically studying and don't have time to go through everything, head to section 8, which is a step-by-step answer guide for the April 2019 exam.

Contents

0 Prerequisites, LP refresh

You need to know what the simplex method is and be able to do it. The rest should be clear. I will add info on how-to-simplex later but this doesn't have a huge priority right now.

1 Transportation and Assignment Problems

Linear programming is all about maximizing the reward / minimizing the cost within the constraint boundaries. The Transportation Problem is a type of linear program typically used to determine how to optimally transport goods (it has many other uses). The Assignment Problem is a subset of this, used to assign people tasks. Solving these problems can be done with the Transportation Simplex Method. Just remember: ***sum of supplies = sum of demands***

1.1 What you need to know

- *Golden rule: Sum of Supplies = Sum of Demands* - if they are not equal, compensate with dummy values
- *Big M values* - these are penalties. M represents a massive cost for supplying demand point A with supply B, making that operation infeasible
- *Dummy values* - these even out the supply and demand
- *Working with minimum & maximum values for supply & demand*
- *Transportation simplex (pivoting)* - Solving a transportation problem

1.2 The Transportation Problem

Imagine you want to transport goods from a factory to a customer. The factory's output and the customer's input are constrained by the factory's *supply* and the customer's *demand*. A factory can produce products at different costs and in different quantities depending on the day; the demand of the customer may change too.

1.3 The Assignment Problem

Steven's example for this is so good and encapsulates everything so well that you should look at that. I will add an example here later for airline scheduling, something that might come on the exam.

2 Network Optimization Models

3 Integer (Linear) Programming

3.1 Cutting Planes and Minimal Covers - Easy Points!

**There is usually a question on the exam about this and it is relatively easy to get full marks for it.*
Cutting Planes are constraints added to the LP to reduce the feasible region (search space) of the LP-relaxation = make new simple constraints using the constraints you already have; makes everything easier

Steven's example (ILP Solving lecture p34): $6x_1 + 3x_2 + 5x_3 + 2x_4 \leq 10$ (all vars binary)
This implies: $x_1 + x_2 + x_4 \leq 2$

But how does this work? Well, it's all about finding out what is and what isn't possible. We are trying to find the combination that invalidates the constraint:

If $x_1 = 1, x_3 = 1$, the constraint is not satisfied (total is at least 11, regardless of other constraint values).
Hence, x_1 & x_3 can't both be equal to 1. Simplifying all of this $x_1 + x_3 \leq 1$. This is a ***cutting plane***.

If $x_2 = 1, x_3 = 1, x_4 = 1$, the total is at least 10, and the constraint is satisfied.
If however $x_1 = 1, x_2 = 1, x_4 = 1$, then the total is at least 11, and so the constraint is not satisfied.

Hence, x_1, x_2, x_4 can't all be equal to 1, so the ***second cutting plane*** would be $x_1 + x_2 + x_4 \leq 2$.

Note that these are all the cutting planes we need. $x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1$ *also invalidates the constraint, but the other cutting planes (remember that they are constraints) already take care of this.*

Now we have our cutting planes, so what is meant by a ***Minimal Cover***? A minimal cover is a set of variables which, assuming the other variables not in the set are set to zero, will satisfy the constraint if and only if at least one variable is set to zero. If you calculated your cutting planes correctly, all you will need to do is put the variables from the cutting planes into a set with curly brackets.
Min covers for above example: $\{x_1, x_3\}$ & $\{x_1, x_2, x_4\}$.

3.2 Graph coloring & other problems - Tricky

See 8.1 from the exam.

4 Nonlinear Programming

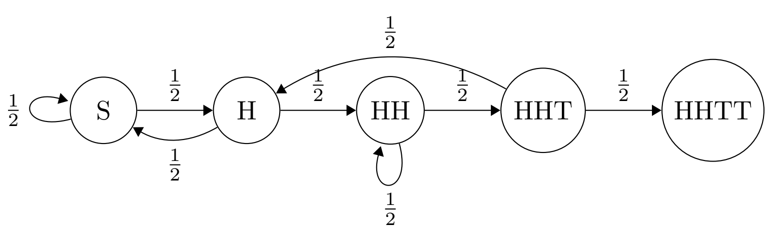
So what's the difference between concave and convex functions? Which ones are which? Con<u>cave</u> functions are the ones that look like the entrance of a <u>cave</u> $\cap$. If you don't know what a cave entrance looks like I would advise you to check out the many caves of Maastricht. A concave function has a set maximum, and a convex function has a set minimum.

5 Markov Chains

A Markov chain is a stochastic model describing a sequence of possible events in which the probability of each event depends only on the state attained in the previous event. - Wikipedia

5.1 Coin Flipping

Coin flipping itself won't be on the exam; it is however strongly recommended to learn about it since it is the simplest form of Markov chain.
Coins have two sides, heads (H) and tails (T); $P(H) = 0.5$ and $P(T) = 0.5$. The question we want to know is: How many flips does it take to reach a certain state? We can model this as a Markov chain, where each state in the chain represents the pattern after a flip. Take the following example for the process of flipping coins to get the order $HHTT$.



S represents the start of the chain, when we have not yet flipped the coin. Each time you move to a state (ie. follow an arrow), you add 1 to your number of coin flips. The probability of taking a step along the path is $\frac{1}{2}$ in all cases (H & T). We only include states that will lead to the goal of getting $HHTT$. If you get H after getting HHT, you'll be back to state H. The Markov model makes the process clear, but ***how do you calculate the number of coin flips required to reach the goal?***

E_x represents the expected number of steps to reach a goal state y from state x (so E_y=0). Using the above example of $HHTT$, the method for calculating the expected number of coin flips **from start,** E_S, is as follows: (first write the equations on the left starting from E_S, then do the working out on the right starting from the goal state)

$E_S = 1 + \frac{1}{2}E_S + \frac{1}{2}E_H$
$E_H = 1 + \frac{1}{2}E_S + \frac{1}{2}E_{HH}$
$E_{HH} = 1 + \frac{1}{2}E_{HH} + \frac{1}{2}E_{HHT}$
$E_{HHT} = 1 + \frac{1}{2}E_H + \frac{1}{2}E_{HHTT}$

5. $E_S = 1 + \frac{5}{3} + \frac{1}{2}E_S + \frac{2}{6}E_S \rightarrow \mathbf{E_S = 16}$
4. $E_H = 1 + \frac{1}{2}E_S + (\frac{3}{2} + \frac{1}{4}E_H) \rightarrow E_H = \frac{10}{3} + \frac{2}{3}E_S$
3. $E_{HH} = 1 + \frac{1}{2} + \frac{1}{4}E_H + \frac{1}{2}E_{HH} \rightarrow E_{HH} = 3 + \frac{1}{2}E_H$
2. $E_{HHT} = 1 + \frac{1}{2}E_H + \frac{1}{2} \times 0 \rightarrow$ ...and sub above.

$E_{HHTT} = 0$ **1. Start here:** E_{HHTT} is set to zero, substitute into equation above.

A system like the one above can be

This is a transition matrix -

6 Queueing Theory

7 Markov Decision Processes - INCOMPLETE!!

8 April 2019 Exam questions & solutions

8.1 Integer Linear Programming

a. (*4 points*) You are given an undirected graph $G = (V, E)$. We say that G is *3-colourable* if it is possible to assign the colours red, blue and green to the vertices V of G such that no two adjacent vertices have the same colour. Write an ILP to determine whether G is 3-colourable.

We introduce a binary variable $x_{v,c}$ for each $v \in V$, where $c \in \{r, g, b\}$. The objective function is to minimize the sum of colors of vertices in the graph, $\sum_{c\in\{r,g,b\}} W_c$, where W_c is true (equal to 1) when at least one vertex $v \in V$ is colored with color $c \in \{r, g, b\}$. The constraints are as follows: $x_{u,c} + x_{v,c} \leq 1, \forall u, v \in E, c \in \{r, g, b\}$, ensuring no two neighboring vertices share the same color, and $x_{v,r} + x_{v,g} + x_{v,b} = 1$, so that each vertex has exactly one color.

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b. (*4 points*) We are given an ILP which contains a (not necessarily integer) variable x_a. We want to ensure that at all times the variable falls in the range $(10 \leq x_a \leq 30)$ or $(55 \leq x_a \leq 130)$. Add constraints to make this possible.

The first step is to write down the hard truths. x_a will never be smaller than 10 and will never be greater than 130. How do you deal with the other values? → By using a binary value y and a big M, representing a huge number.
$x_a \geq 10$
$x_a \leq 130$
$x_a \leq 30 + My$, *where* $y = \{0, 1\}$
$x_a \geq 55 - M(1 - y)$

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c. (*4 points*) We are given an ILP where x_a, x_b and x_c are binary variables. Add constraints to ensure that, at all times, binary variable x_d is equal to 0 if all three of x_a, x_b, x_c have value 1, but otherwise (i.e. at least one of x_a, x_b, x_c has value 0) x_d is equal to 1.

Break these things down into logical statements, but remember that we have to arrange these into inequalities. This is a case where the *either-or* technique can be applied, as we have two distinct possibilities, $x_a + x_b + x_c \geq 3, x_d = 0$ and $x_a + x_b + x_c \leq 2, x_d = 1$. Now we have to incorporate x_d into the inequalities, which can be done using big-M penalties. $x_a + x_b + x_c \geq 3 - M(x_d)$ and $x_a + x_b + x_c \leq 2 + M(1 - x_d)$.

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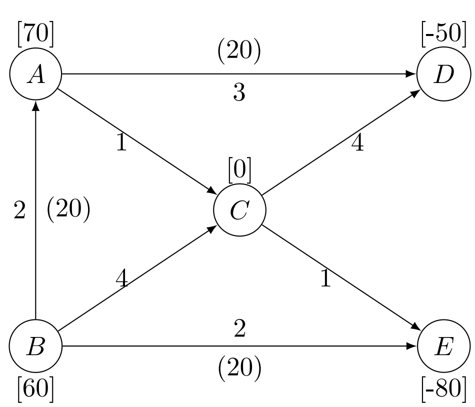
d. (*3 points*) During branch and bound for a *maximization* ILP, it can happen that (when examining a particular node of the search tree) the linear programming relaxation for that node returns a fractional solution with objective function value *strictly larger* than that achieved by the current incumbent. Explain in your own words what the branch and bound algorithm does next.

It's Friday and the sun's out. I'll answer this question another time.

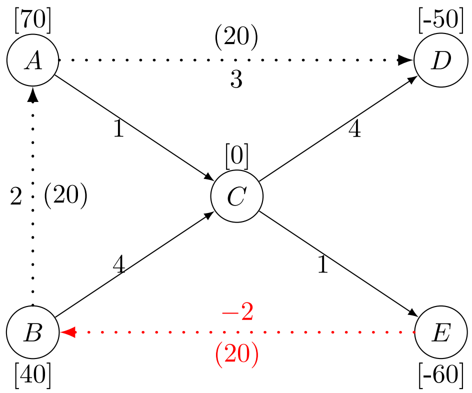
8.2 Network Optimization

Consider the Minimum Cost Flow problem shown below, where the numbers in square brackets denote the supplies and demands; the numbers without brackets on the arcs are the costs per unit flow shipped through the arcs. Arcs $B \to A, A \to D$ and $B \to E$ each have a capacity of 20 units and all other arcs have unbounded capacity.



a. Find an initial BFS by solving the feasible spanning tree with basic arcs $A \to C, B \to C, C \to D$ and $C \to E$. The nonbasic arc $E \to B$ is a *reverse* arc. What is the cost of this initial BFS?

Before we can do anything else, we have to make sure we account for the reversed NB arc $E \to B$, with a capacity of 20. This means E's demand drops to 60 and B's supply drops to 40. We are told the basic arcs are the ones connected to C. From there we do our calculations on the arcs:



$A \to C$:	70 supply	$\times$ 1 cost	$= 70$
$B \to C$:	40 supply	$\times$ 4 cost	$= 160$
$C \to D$:	50 demand	$\times$ 4 cost	$= 200$
$C \to E$:	60 demand	$\times$ 1 cost	$= 60$
$E \to B$:	20 reversed arc	$\times$ 2 cost	$= 40$

The sum is $70 + 160 + 200 + 60 + 40 = 530$, which is the cost of our initial BFS.

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b. Starting from your answer to (a) apply **one** iteration of the Network Simplex method. At this point indicate the cost of the current BFS and the flows in its arcs. Is it optimal?

We need to pivot: the question is, where do we start? What is the entering basic variable?

$\xrightarrow{\textit{Very Important}}$ ***The BFS is a tree. By adding a non-basic variable, we are effectively creating a cycle. In order to see which non-basic variable we should choose as the entering basic variable, we need to calculate the greatest cost saving. We do this by adding the costs of the edges in the cycle. The cost of every edge in the same direction (clockwise/anticlockwise) as the proposed entering basic variable is added to the total, and the cost of each edge in the other direction is subtracted from the total.***

$AD = 3 - 4 - 1 = -2 \leftarrow$ This is the most negative value. It will enter.

$BA = 2 + 1 - 4 = -1$

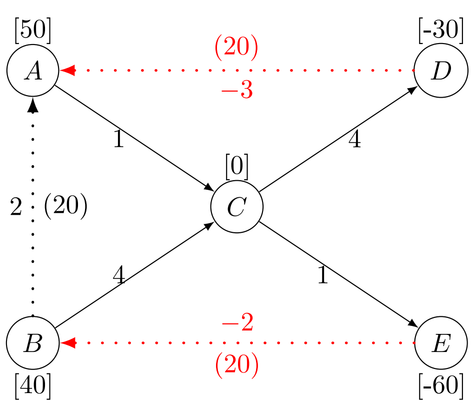
$EB = -2 + 4 + 1 = 3$

So AD will enter the basis. Which basic variable will leave? The leaving basic variable is the variable that reaches its upper capacity limit or its lower limit (0 flow passing through). So if we add the maximum 20 units of flow to AD, does any other basic variable hit its upper limit or hit 0? **No.** So AD leaves again and is flipped to become DA with cost -3; A's supply drops to 50 and D's demand drops to 30. So that's **one** iteration of the Network Simplex method. Now we still have to calculate the costs and the flows in the arcs, and say if it's optimal.

$BA = 2 + 1 - 4 = -1 \leftarrow$ Negative, not optimal!

$DA = -3 + 1 + 4 = 2$

$EB = -2 + 4 + 1 = 3$



$A \to C$:	50 supply	$\times$ 1 cost	$= 50$
$B \to C$:	40 supply	$\times$ 4 cost	$= 160$
$C \to D$:	30 demand	$\times$ 4 cost	$= 120$
$C \to E$:	60 demand	$\times$ 1 cost	$= 60$
$D \to A$:	20 reversed arc	$\times$ 3 cost	$= 60$
$E \to B$:	20 reversed arc	$\times$ 2 cost	$= 40$

The sum is $50 + 160 + 120 + 60 + 60 + 40 = 490$, which is the cost of our current BFS. It is not optimaln as there is still a negative non-basic variable (BA).

8.3 Non-linear Programming

Consider the following non-linear programming problem. Note that it is a *minimization* problem!

$$\begin{aligned}&\text{Minimize } f(x) = -x_1 - 2x_2 + x_2^3\\&\text{subject to}\\&x_1 + x_2 \le 1\\&\text{where } x_1, x_2 \ge 0\end{aligned}$$

a. Verify that this is a convex programming problem.

In order to verify that this is a convex programming problem, we need to first convert the objective function to a *maximization* problem, so the new objective function will be $f(x) = x_1 + 2x_2 - x_2^3$. Now we can calculate the derivatives.

$$\frac{df}{dx_1} = 1 \qquad \frac{d^2f}{dx_1^2} = 0 \le 0$$

$$\frac{df}{dx_2} = -3x_2^2 + 2 \qquad \frac{d^2f}{dx_2^2} = -6x_2 \le 0$$

$$\frac{d^2f}{dx_1, x_2} = 0 \qquad \left(\frac{d^2f}{dx_1^2} \times \frac{d^2f}{dx_2^2}\right) - \left(\frac{d^2f}{dx_1, x_2}\right)^2 = 0 \ge 0$$

Hence, f is concave, so it's a convex program! *(This may seem confusing; just remember this rule, and that you need to flip minimization problems to maximization problems.)*

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b. Write out the Karush Kuhn Tucker (KKT) conditions for this problem and then use these to verify that there is an optimal solution where $x_2 = \frac{1}{\sqrt{3}}$.

First, we need to figure out the KKT conditions. See the formula sheet at the end! It's best to start with a table of derivatives. f refers to the objective function, g_i refers to the constraints. u_i refers to variables which must hold throughout the equations; i refers to the constraint number (same i as g). KKT conditions are on the right.

$\frac{\delta\text{column}}{\delta\text{row}}$	f	g_1
x_1	1	1
x_2	$-3x_2^2 + 2$	1

$$\begin{aligned}
\mathbf{x_1}: \; & 1 - (u_1) \le 0 && (1)\\
& x_1(1 - (u_1)) = 0 && (2)\\
\mathbf{x_2}: \; & -3x_2^2 + 2 - (u_1) \le 0 && (3)\\
& x_2(-3x_2^2 + 2 - (u_1)) = 0 && (4)\\
\mathbf{g_1}: \; & x_1 + x_2 - 1 \le 0 && (5)\\
& u_1(x_1 + x_2 - 1) = 0 && (6)\\
& \text{all } x_i \ge 0 && (7)\\
& \text{all } u_i \ge 0 && (8)
\end{aligned}$$

So now we need to verify that there is indeed an optimal solution where $x_2 = \frac{1}{\sqrt{3}}$ (substitute this). (*Note: A lot of these conditions look very similar; each second equation is the first equation turned into an equality. Use this to your advantage.*)

First look at **(4)**; since $x_2 \ne 0$, the rest of (4), between brackets, must be zero to satisfy the equality. This means **(3)** should be equal to 0. If we substitute x_2 for $\frac{1}{\sqrt{3}}$, we get $-3 \times \frac{1}{3} + 2 - (u_1) \le 0$. This means $u_1 = 1$. So far, this means (3), (4) and **(8)** are satisfied. *Remember that we need to satisfy all conditions for optimality.* So let's move on to **(1)**; it's satisfied, since $0 \le 0$. This immediately means **(2)** is also satisfied. At this point, x_1 can be whatever we want. From **(6)**, we can see that **(5)** evaluates to 0. Hence, $x_1 = \frac{\sqrt{3}-1}{\sqrt{3}}$. **(7)** is also satisfied. With this, we can conclude that an optimal solution exists where $x_1 = \frac{\sqrt{3}-1}{\sqrt{3}}$, $x_2 = \frac{1}{\sqrt{3}}$, $u_1 = 1$.

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c. Suppose we wished to optimize the function $f(x)$ but without any constraints on the allowed values of x_1 and x_2. We could use *gradient search* for this. Each iteration of gradient search requires us to compute the global optimum of an unconstrainted single variable function $h(t)$. What is the function $h(t)$ in the first iteration of gradient search if we take $(x_1, x_2) = (1, 1)$ as our starting point? You do not need to simplify or optimize the function $h(t)$ or to proceed further.

An answer to this is on its way. If you have an answer already, please send it over so I can check it and put it in to help other students.

8.4 Transportation & Assignment Problems

a. There are *three* university courses that need to be taught, and each course requires a specific number of teaching hours. There are *four* teachers, who have to teach these courses. Teachers can split courses up between them, and the cost to the university for an hour of teaching depends on the specific course and teacher (because of different experience levels). The goal is to allocate the teaching hours at minimum total cost.

The course/teacher costs are shown in the table below. Some teachers have upper bounds on the number of hours they want to teach, some have lower bounds, some have both. Teachers do not have to use all their available teaching hours, as long as they satisfy their lower bound.Note also that some teachers refuse to teach some courses, represented by a "-" in the table. Explain how you can model this cost-minimization problem as a Transportation Problem. You do not need to solve the problem.

		Teacher					
		T1	T2	T3	T4		
Course	C1	3	6	4	7	60	Hours that
	C2	2	-	6	8	40	need to be taught
	C3	8	5	2	3	60	
Min teaching hours		50	0	30	40		
Max teaching hours		50	40	80	INFTY		

Each step to solving the problem is represented by a color in the table below. First thing's first - $C2, T2$ is not feasible, so we need to turn this into a big M.
Following this, we need to figure out what to do with supply and demand. Demand is $60 + 40 + 60 = 160$. To figure out what the maximum is for $T4$, we need to calculate the minimum supply without $T4$, so $50 + 0 + 30 = 80$, so $T4$'s maximum is $160 - 80 = 80$.
The next step is balancing supply & demand. Max supply - demand $= (50 + 40 + 80 + 80) - (60 + 40 + 60) = 90$, so we need a dummy demand variable ($C4$) to take in the 90 hours of extra supply.
We are not done yet as we have only calculated the upper bound. Now we need to calculate the lower bound for each supply point, taking account of the dummy variable. $T1$ wants to work exactly 50 hours; they will not be happy if these are dummy hours - lower bound is 50.
$T2$ has a lower bound of 0, so they don't mind working dummy hours (the dummy has no cost) - their supply is kept at 40.
$T3$ and $T4$ need to be split up, as they have nonzero lower bounds, and the difference between lower and upper bounds is smaller than the dummy demand. $T3A$ will have a minimum of 30 hours, with big M for the cost of dummy hours; $T3B$ will have a maximum of 50 hours $(80 - 30)$ and will allow dummy hours. Likewise, $T4A$ will have a minimum of 40 hours, with a big M for the dummy; $T4B$ will have a maximum of 50 hours $(80 - 30)$ and will allow dummy hours.

	$T1$	$T2$	$T3A$	$T3B$	$T4A$	$T4B$	
$C1$	3	6	4	4	7	7	60
$C2$	2	M	6	6	8	8	40
$C3$	8	5	2	2	3	3	60
$C4(D)$	M	0	M	0	M	0	90
	50	40	30	50	40	40	

b. Consider the Transportation table below. There are 3 supply points (with supplies 12, 8 and 10) and 3 demand points (with demands 11, 5, 14). The table shows the cost of shipping one unit from each supply point to each demand point. We wish to minimize the shipping costs.
Use the Transportation Simplex method to compute the optimum solution for this problem. **Use as starting point the following basic feasible solution:** $(S_1, D_1) = 11, (S_2, D_2) = 1, (S_2, D_2) = 4, (S_2, D_3) = 4$ and $(S_3, D_3) = 10$.

				Supply	
	9	7	8	12	S1
	3	5	1	8	S2
	4	3	9	10	S3
Demand	11	5	14		
	D1	D2	D3		

Explain how you know you have reached optimality and be sure to explicitly state the cost of the optimal solution. If you have done your calculations correctly you will require only a very small number of pivots.

We've been given a BFS. First, draw a tableau like the one below. Start by filling in the supplies and demands, and then write down the costs in the small inner-cells. If we were asked to make a BFS, we'd need to use a very important step: the North-West Corner Rule. Start at the left top corner; fill in the NW cell with the highest value possible ($max(supply, demand)$). So here, we'd fill in 11; since we've satisfied the demand in that column but not yet the row's supply, we will move on to the next column, same row. Repeat until all supplies and demands are satisfied.

Anyway, here we're given a BFS already, so we just fill that in (done in red), together with the sum of the products of the costs per node in the BFS ($11 \times 9 + 1 \times 7 + 4 \times 5 + 4 \times 1 + 10 \times 9 = 220$).

Now we need to calculate the $\boldsymbol{u_i}$ and $\boldsymbol{v_j}$ values. The idea is that for the basic variables, $c_{ij} - u_i - v_j = 0$. Let's set $u_2 = 0$, since there are two basic variables in that row. If that's the case, $v_3 = 1$, so that for (S_2, D_3), $1 - 0 - 1 = 0$. This means that $u_3 = 8$, so that (S_3, D_3) is balanced. Do this for all the basic variables until you have filled in all the values for $\boldsymbol{u_i}$ and $\boldsymbol{v_j}$

Now you can enter in the values for the non-basic variables $= c_{ij} - u_i - v_j$.

	1	2	3	Supply	u_i
1	cost 9; 11	cost 7; 1	cost 8; 5	12	2
2	cost 3; -4	cost 5; 4	cost 1; 4	8	0
3	cost 4; -11	cost 3; -10	cost 9; 10	10	8
Demand	11	5	14	220	
v_j	7	5	1		

As can be seen above, our table is not optimal since we have negative non-basic variables. *It's pivoting time!* $\rightarrow$ choose the most negative value in the table (S_3, D_1 here); this will become the *entering basic variable*. Please see my step-by-step YouTube video of this done using IORTutorial: `https://youtu.be/cXj6nmC3gDc`.

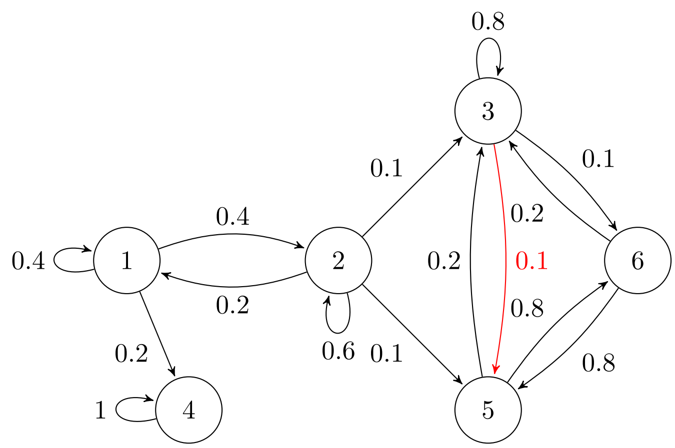
8.5 Markov Chains

(15 points) We examine a discrete Markov chain with state space $\{1, 2, 3, 4, 5, 6\}$ and transition matrix P given by

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} & P_{15} & P_{16} \\ P_{21} & P_{22} & P_{23} & P_{24} & P_{25} & P_{26} \\ P_{31} & P_{32} & P_{33} & P_{34} & P_{35} & P_{36} \\ P_{41} & P_{42} & P_{43} & P_{44} & P_{45} & P_{46} \\ P_{51} & P_{52} & P_{53} & P_{54} & P_{55} & P_{56} \\ P_{61} & P_{62} & P_{63} & P_{64} & P_{65} & P_{66} \end{pmatrix} = \begin{pmatrix} 0.4 & 0.4 & 0 & 0.2 & 0 & 0 \\ 0.2 & 0.6 & 0.1 & 0 & 0.1 & 0 \\ 0 & 0 & 0.8 & 0 & 0.1 & 0.1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0.8 \\ 0 & 0 & 0.2 & 0 & 0.8 & 0 \end{pmatrix}$$

a. Calculate, for initial state 1, the probability of ever arriving in state 4, and also calculate the probability of ever arriving at state 6.

We want to go from 1 to 4. The best way to represent this is to draw it, also in an exam. (Red arrow to show which probability belongs where).



It's clear from the Markov chain above that there is no way to reach state 4 from states $3, 5, 6$. This means their probability is zero $P_3 = P_5 = P_6 = 0$. The probability of $P_4 = 1$, since it is the goal state. (First do the calculations on the left top to bottom, then simplify on the right from the bottom upwards).

$P_1 = 0.4P_1 + 0.4P_2 + 0.2$ $\quad$ $P_1 = 0.4P_1 + 0.2P_1 + 0.2 \rightarrow \mathbf{P_1 = 0.5}$

$P_2 = 0.2P_1 + 0.6P_2$ $\quad$ $0.4P_2 = 0.2P_1 \rightarrow P_2 = 0.5P_1$

• And so we see that the probability (P_1) of ever reaching 4 from 1 is 0.5.

Now we need to calculate the probability of ever getting to 6 from 1. We know 4 will never lead us to 6 so $P_4 = 0$. $P_6 = 1$, since it is the goal state. Again, start left from P_1 to P_5, then simplify bottom-to-top from P_5 on the right.

$P_1 = 0.4P_1 + 0.4P_2$ $\quad$ $P_1 = 0.4P_1 + 0.2P_1 + 0.2 \rightarrow 0.4P_1 = 0.2 \rightarrow \mathbf{P_1 = 0.5}$

$P_2 = 0.2P_1 + 0.6P_2 + 0.1P_3 + 0.1P_5$ $\quad$ $P_2 = 0.2P_1 + 0.6P_2 + 0.1 + 0.1 \rightarrow 0.4P_2 = 0.2P_1 + 0.2 \rightarrow P_2 = 0.5P_1 + 0.5$

$P_3 = 0.8P_3 + 0.1P_5 + 0.1$ $\quad$ $P_3 = 0.8P_3 + 0.02P3 + 0.08 + 0.1 \rightarrow 0.18P_3 = 0.18 \rightarrow P_3 = 1$

$P_5 = 0.2P_3 + 0.8$

• And so we see that the probability for getting from 1 to 6 is 0.5. 4 is an absorbing state, we're never going to 6 from there, so the probability for getting to 6 from 4 is 0. In the above example, you might wonder why P_3 and P_5 are equal to 1. The reason for this is because $3, 5, 6$ are all recurrent states and as a set, are communicating; this means that for any of these states, it is possible that these states will be revisited in the future. What this means is that if you visit 3 or 5, you will eventually end up at 6; so the probability of ever arriving at 6 from either 3 or 5 is 1.

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b. Calculate the unique stationary distribution π, that corresponds to initial state 1.

What do we know so far? We know states 1 and 2 are transient: eventually, we will leave these states and not return. Therefore, states π_1 and π_2 are set to 0. π_4 is equal to the probability of getting from 1 to 4; hence, $\pi_4 = 0.5$. Calculating π_3, π_5, π_6 is a little more difficult, since these are all recurrent communicating states. $\sum \pi_i = 1$, so $\pi_3 + \pi_5 + \pi_6 = 1 - 0.5 = 0.5$, which we also know from part **a.**
When calculating this, only look at what's going out from a state, and multiply this by π_i of that state i. So, $0.8\pi_6 = 0.8\pi_5$. This means $\pi_5 = \pi_6$. $0.1\pi_3 = 0.2\pi_5$, so $\pi_3 = 2\pi_5$. In words, this means you will visit π_3 twice as often as π_5 or π_6. Out of a total of 1 time spent, you will be in π_3 for $\frac{1}{2}$ the time, and in π_5 or π_6 each for $\frac{1}{4}$ of the time. So, $(\pi_3, \pi_5, \pi_6) = 0.5(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. Hence, $\pi = (0, 0, 0.25, 0.5, 0.125, 0.125)$. Always make sure it adds up to one!

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c. Calculate, for initial state 1 and for each transient state s, the expected number of stages that the process will be in that particular state s.

Easy points! The transient states here are 1 and 2. So all we need to focus on are the probabilities in the left top corner of the transition matrix. $P_T = \begin{pmatrix} 0.4 & 0.4 \\ 0.2 & 0.6 \end{pmatrix}$. Our goal is to find S, calculated by $S = (I - P_T)^{-1}$.

$$S = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.4 & 0.4 \\ 0.2 & 0.6 \end{pmatrix} \right)^{-1} = \begin{pmatrix} 0.6 & -0.4 \\ -0.2 & 0.4 \end{pmatrix}^{-1} = \frac{1}{0.6\times0.4-0.4\times0.2} \begin{pmatrix} 0.4 & 0.4 \\ 0.2 & 0.6 \end{pmatrix} = \begin{pmatrix} 2.5 & 2.5 \\ 1.25 & 3.75 \end{pmatrix}$$

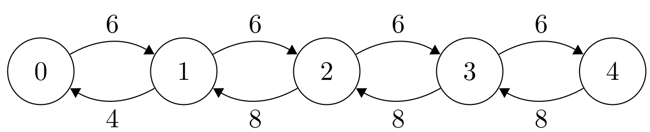
This means that, starting from stage 1, the expected number of stages that the process will be in state 1 is 2.5 and that, also starting from stage 1, that the expected number of stages that the process will be in state 2 is also 2.5.

8.6 Queuing Models

(15 points) At a small first aid center, patients arrive according to a Poisson-process with an average of 6 patients per hour. There are 2 medical doctors available to examine the patients before they either leave or continue into hospital care. The examinations are exponentially distributed with an average of 15 minutes per patient. If the doctors are both busy, any new patient has to wait in one of the 2 remaining available private examination rooms (1 patient per room maximum). During the time intervals that these rooms are all occupied, any new patients are immediately directed to a different first aid center slightly further away. So there are at most 4 patients in the system, 2 waiting and 2 being examined.

a. Give the rate diagram (the graph with transition rates) and calculate the steady state probabilities (the limit probabilities) π_i for all states i.

Remember: We are given rates and times - these are two different concepts. We need to make sure that these are standardized in our calculations. Rate in = 6 patients per hour (60 minutes). Each examination is distributed with an average of 15 minutes (so the rate out = 4 *per doctor*). If we have 1 patient in the system, the rate out is 4 per hour. If we have 2 patients in the system, our rate out is 8 per hour. After that, we have no more doctors to deal with patients, so our rate out stays fixed at 8 per hour. *The states in the chain below represent the patients in the system, not the doctors.* Additional intuition: if the system is full, we do not accept patients. We need to go back to having a maximum of 3 patients in the system before being able to accept new patients.



So now we need to calculate the steady state probabilities π_i for all states i. **Rate In = Rate Out**. We need to set everything in terms of π_4, since π_4 (for now) = 1.

$$\begin{aligned} \text{Rate in} &= \text{Rate out} \\ 4\pi_1 &= 6\pi_0 \\ 8\pi_2 &= 6\pi_1 \\ 8\pi_3 &= 6\pi_2 \\ 8\pi_4 &= 6\pi_3 \end{aligned}$$

Putting everything in terms of π_4 (*Reversed order from 3 to 0*):

$$\begin{aligned} \pi_3 &= \frac{8}{6}\pi_4 \\ \pi_2 &= \frac{16}{9}\pi_4 \\ \pi_1 &= \frac{64}{27}\pi_4 \\ \pi_0 &= \frac{128}{81}\pi_4 \end{aligned}$$

So this means $\pi = \pi_4(\frac{128}{81}, \frac{64}{27}, \frac{16}{9}, \frac{8}{6}, 1)$. Now we need to give all these values the same denominator. $\pi = \pi_4(\frac{128}{81}, \frac{192}{81}, \frac{144}{81}, \frac{108}{81}, \frac{81}{81})$. In order to get π, we need to remember that $\sum \pi_i = 1$. Add all the numerators together, this will become the new denominator. $\pi = (\frac{128}{653}, \frac{192}{653}, \frac{144}{653}, \frac{108}{653}, \frac{81}{653})$ is our answer.

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b. What is the expected number of patients in this first aid center (patients that are either waiting or being examined) at an arbitrary point in time?

This works as follows: multiply the number of agents (doctors in this case) working by that number's steady state probability calculated above. Note that you can't have more than 2 agents working, so do not go beyond 2. $L = 0 \times \frac{128}{653} + 1 \times \frac{192}{653} + 2 \times \frac{144}{653} + 2 \times \frac{108}{653} + 2 \times \frac{81}{653} = \frac{858}{653} \approx 1.3$ customers in the system at an arbitrary point in time.

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c. Suppose that at 9:00 in the morning the two doctors have just checked in and are ready for patients, while no one is waiting yet. From that moment on, how long will it take (on average) until both doctors are occupied?

Restating the question: if starting in state 0, how long will it take before you reach state 2? This means anything beyond state 2 is not important. E_i corresponds to the expected number of steps to reach a goal state from state i. Like in the coin flipping example in section 5, we will set the expected steps to the goal state from the goal state (E_2) to 0. Keep the following in mind: We are being asked *how long* it will take (i.e. time in minutes). Remember that 10 actions per hour (**rate**, so $\frac{10}{60}$) is the same as saying every 6 minutes (**time**, so $\frac{60}{10}$). The formula is thus: $E_i = \frac{\text{time unit}}{\text{total outgoing}} + \frac{\text{out}_{\text{to } i-1}}{\text{total outgoing}} \times E_{i-1} + \frac{\text{out}_{\text{to } i+1}}{\text{total outgoing}} \times E_{i+1}$. Remember that we will set E_2 to 0 since 2 is the goal state.

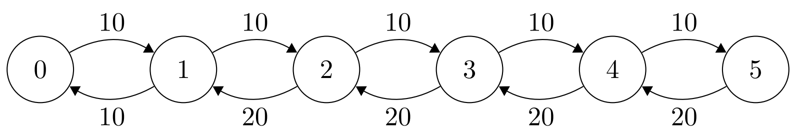
$$E_0 = \frac{60}{6} + E_1$$
$$E_1 = \frac{60}{10} + \frac{4}{10}E_0 + \frac{6}{10}E_2$$
$$E_2 = 0$$
$$E_1 = \frac{60}{10} + \frac{4}{10}E_0$$
$$E_0 = \frac{60}{6} + \frac{60}{10} + \frac{4}{10}E_0 \rightarrow \frac{6}{10}E_0 = 10 + 6 \rightarrow E_0 = \frac{160}{6} = 26.666... \approx 27 \text{ minutes}$$

8.7 Queuing Models April 2018

Just realized this question is from April 2018, not 2019! I will add the answers to question 6 from 2019 soon.(15 points) At a small call center, calls arrive according to a Poisson-process with an average of 10 calls per hour. There are 2 agents available to answer these calls. The length of each call is exponentially distributed with an average of 6 minutes. If the agents are both busy, any new caller is put on hold, provided that the waiting line facility has at least one of its 3 available slots empty. During the intervals that these slots are all occupied, any new callers are simply lost and do not keep calling. So there are at most 5 customers in the system, 3 waiting and 2 being served.

a. Give the rate diagram (the graph with transition rates) and calculate the steady state probabilities (the limit probabilities) π_i for all states i.

Remember: We are given rates and times - these are two different concepts. We need to make sure that these are standardized in our calculations. Rate in = 10 calls per hour (60 minutes). Each call is distributed with an average of 6 minutes (so the rate out = 10 *per agent*). If we have 1 customer in the system, the rate out is 10 per hour. If we have 2 customers in the system, our rate out is 20 per hour. After that, we have no more agents to deal with customers, so our rate out stays fixed at 20 per hour. *The states in the chain below represent the customers in the system, not the agents.* Additional intuition: if the system is full, we do not accept callers. We need to go back to having a maximum of 4 customers in the system before being able to accept calls.



S represents something I guess.