Exam **Discrete Mathematics (resit), Tuesday January 26th 2016, 14:00h-17:00h**
Bachelor program Knowledge Engineering, year 1
Tapijnkazerne, building Z

This exam consist of 8 problems, totalling 100 points, which you can solve in any order you like. Be sure to motivate all your answers with calculations/proofs/arguments.
Remember that $\mathbb{N} = \{1, 2, 3, \ldots\}$. Good luck!

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1. **(10 points)** Fill in the truth table for the following logical proposition.
   - $((p \Leftrightarrow r) \Rightarrow \neg q) \wedge (p \Rightarrow (q \vee \neg r))$

2. **(15 points)** Use induction to prove the following statements.
   (a) For all integers $n \geq 0$,
   $$\sum_{i=0}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$
   (b) It is possible to make all coin totals of 60 cents or more using only 6 cent and 11 cent coins.

3. **(15 points)** Prove or disprove the following statements. (Recall that A^c denotes the *complement* of A and $\mathbb{P}(A)$ the *powerset* of A).
   (a) For all sets A, B and C, $((A \cap B)^c \cup C)^c = ((A \cap B) \setminus (A \cap C))$,
   (b) For all sets A and B, $(\mathbb{P}(A \setminus B) \cup \mathbb{P}(A \cap B) \cup \mathbb{P}(B \setminus A)) = (\mathbb{P}(A) \cup \mathbb{P}(B))$.

4. **(15 points)** This question is about *relations*.
   (a) Let R be the relation on $\mathbb{N}$ defined as follows: xRy means "$x \leq 2y - 1$". Is R reflexive? Symmetric? Transitive? Anti-symmetric? For each of these properties, prove or disprove that it has that property.
   (b) Let R be a relation on $\mathbb{Z}$ defined as follows: xRy means "$x - y$ is natural". Is R reflexive? Symmetric? Transitive? Anti-symmetric? For each of these properties, prove or disprove that it has that property.
   (c) Let $A = \{0, 1, 2\}$. Does there exist a relation R on A that is *both* an equivalence relation and a partial order? If so, draw such a relation (the relation does not need to have a "real-world" meaning). If not, explain briefly why not.

5. **(15 points)** All the following questions are about *counting*.
   (a) Let A and B be sets such that $|A| = 5$ and $|B| = 7$. How many injective functions are there from A to B?
   (a) An ice-cream shop has 4 different flavours of ice-cream available: mint choc-chip, raspberry, vanilla and chocolate. A group of 9 people enters the shop, and each person orders an ice-cream. How many different choice patterns are possible, assuming order is not important? (So, for example, *"Three mint choc-chip, two raspberry, four vanilla"* would be considered a single pattern.)
   (c) How many different length 6 strings can you make using "0", "1" and "2" symbols, such that each string uses all three symbols (i.e. each string contains at least one "0", at least one "1" and at least one "2")? Note that two strings are considered identical if and only if they are identical in every position. *Hint:* inclusion-exclusion might help you here.
   (d) Let A be a set with 10 elements. How many subsets are there of A which contain *exactly* 4 elements?

6. **(10 points)** Prove or disprove the following statements.

   (a) $(\exists x \in \mathbb{Z})(\forall y \in \mathbb{Z})(x + y \in \mathbb{N})$

   (b) $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})(\frac{x}{y} \notin \mathbb{N})$

   (c) $\neg((\exists x \in \mathbb{Z})(\forall y \in \mathbb{R})(xy > 0))$

   (d) $(\exists x \in \mathbb{N})(\forall y \in \mathbb{Z})(\exists z \in \mathbb{N})(x + y + z \geq -32)$.

7. **(15 points)** This is a question about *functions*.

   (a) Suppose a set A has 5 elements and a set B has 3 elements. How many functions are there from B to A?

   (b) Suppose $A = \{0, 1, 2\}$ and $B = \{-2, -1, 0, 1, 2\}$. Let $g : A \to B$ be defined as $g(x) = 1 - x$. Draw a function $f : B \to A$ such that $(g \circ f)(x)$ behaves as follows: $(g \circ f)(x) = -1$ (if $x < 0$), $(g \circ f)(x) = 0$ (if $x = 0$) and $(g \circ f)(x) = 1$ (if $x > 0$). Recall that $(g \circ f)(x)$ is notationally equivalent to $g(f(x))$.

   (c) Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ and let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be the function defined as follows:

   $$f(x) = \begin{cases} 3x & \text{if } 0 \leq x \leq 7 \\ (x-7)^2 + c & \text{if } x > 7 \end{cases}$$

   There exists only one possible value for the constant c which can make f invertible. What is it? Motivate this by proving that the function, for your choice of c, is invertible and give also f^{-1}.

8. **(5 points)** Let $A = \{1, \{1\}, \emptyset\}$, $B = \{1, \{\emptyset\}\}$ and $C = \{3, 4, 5\}$.

   (a) Write down $\mathbb{P}(A \cap B)$

   (b) Write down all partitions of C.